THEOREMS OF THE ALTERNATIVE FOR COMPLEX LINEAR INEQUALITIES*

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ABSTRACT

The classical transposition theorems of Motzkin, Gordan, Stiemke and others are extended to complex linear inequalities.

Introduction. Consider the system

(1) $Ax = 0, \quad x \in S$

where $A \in C^{m\times n}$ and S is a polyhedral cone in C^n (1). The existence of nontrivial solutions (2) of systems like (1) is studied here in a sequence of *theorems of the alternative*, each listing two systems such as:

(I) $Ax = 0$, x nontrivial vector in S(2)

(II) $A^H y$ nontrivial vector in S^*(2).

exactly one of which is consistent. These theorems have as corollaries the *transposition theorems* (3) and *theorems of the alternative* for linear inequalities, given in the references, in particular the classical theorems of Motzkin [13] [14], Gordan [6] and Stiemke [17].

0. **Notations and preliminaries.**

$C^n[R^n]$ *the n-dimensional complex [real] vector space*

$C^{m\times n}[R^{m\times n}]$ *the mxn complex [real] matrices*

$R^n_+ = \{x \in R^n : x_i \geqq 0 \ (i = 1,\cdots,n)\}$ *the nonnegative orthant in* R^n.

For any $A \in C^{m\times n}$:

$$A^C \text{ the } \textit{conjugate}, \ A^T \text{ the } \textit{transpose}, \ A^H = A^{CT}$$

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(1) See notations in §0.

(2) *Nontrivial* here means nonzero or even that a subvector lies in the (relative) interior of a given cone.

(3) A term explained by the fact that in the real case the system (II) uses the transposed matrix A^T.

For any $x,\ y \in R^n$:

$x \geqq y$ denotes $x_i \geqq y_i$ $(i = 1,\cdots,n)$, $x \geq y$ denotes $x \geqq y$ and $x \neq y$

$x > y$ denotes $x_i > y_i$ $(i = 1,\cdots,n)$

For any $x,\ y \in C^n$: $(x,y) = y^H x$

A nonempty set S in C^n is a

(i) *convex cone* if $S + S \subset S$ and if $\alpha \geqq 0 \Rightarrow \alpha S \subset S$

(ii) *pointed convex cone* if (i) and if $S \cap (-S) = \{0\}$

(iii) *polyhedral cone* if $S = BR^k_+$ for some $B \in C^{nxk}$

For any nonempty set S in C^n let

$S^* = \{y \in C^n : x \in S \Rightarrow \mathrm{Re}(y,x) \geqq 0\}$ the *polar* of S, e.g. [3]

$\mathrm{int}S^* = \{y \in C^n : 0 \neq x \in S \Rightarrow \mathrm{Re}(y,x) > 0\}$ the *interior* of S^*.

S^* is a closed convex cone.

Since S^* coincides with the polar of the smallest closed convex cone containing S, e.g. [3], it suffices to study polars of closed convex cones. Thus for example sets whose polars have interior points are characterized in:

LEMMA 0. *Let S be a closed convex cone in C^n. Then* $\mathrm{int}\,S^* \neq \varnothing$ *if and only if S is pointed.*

Proof. *If*: e.g. [7] Theorem 2.1.

Only if: Suppose S is not pointed, and thus contains a nonzero vector x together with $-x$. Then for any $y \in \mathrm{int}\,S^*$, $\mathrm{Re}(y,x) > 0$ and $\mathrm{Re}(y,-x) > 0$, a contradiction. Therefore $\mathrm{int}\,S^* = \varnothing$. □

Since $S = S^{**}$ if and only if S is a closed convex cone, e.g. [3] Theorem 1.5, it follows that for a closed convex cone S, int S defined by int $S = \{x \in S : 0 \neq y \in S^* \Rightarrow \mathrm{Re}(y,x) > 0\}$, is nonempty if and only if S^* is pointed.

Another result needed below is the following solvability theorem (3.5 of [3]) for polyhedral systems:

THEOREM 0. *Let $A \in C^{mxn}$, $b \in C^m$ and let S be a polyhedral cone in C^n. Then*

$$Ax = b, \quad x \in S$$

is consistent if, and only if,

$$A^H y \in S^* \Rightarrow \mathrm{Re}(b,y) \geqq 0.$$

1. **Results.** All the results that follow are formulated as *theorems of the alternative*, each listing two systems denoted by (I) and (II), exactly one of which has solutions.

The main result is:

THEOREM 1. *Let*

$A_i \in C^{mxn_i}$, $(i = 1,\cdots,4)$ $A_1 \neq 0$, $A_2 \neq 0$, *T a polyhedral cone* in C^m *S_i polyhedral cones in* C^{n_i}, $(i = 1,2,3)$, *S_1 and S_2^* pointed.*

Then exactly one of the following two systems is consistent:

$$\text{(I)} \qquad \sum_{i=1}^{4} A_i x_i \in T, \quad \left\{ \begin{matrix} 0 \neq x_1 \in S_1, & x_2 \in S_2 \\ \text{or} & \\ x_1 \in S_1, & x_2 \in \operatorname{int} S_2 \end{matrix} \right\}, \quad x_3 \in S_3$$

$$\text{(II)} \qquad y \in -T^*, \quad A_1^H y \in \operatorname{int} S_1^*, \quad 0 \neq A_2^H y \in S_2^*, \quad A_3^H y \in S_3^*, \quad A_4^H y = 0$$

Proof. (I) and (II) cannot have both solutions, for then

$$0 \geqq \operatorname{Re}\left(\sum_{i=1}^{4} A_i x_i, y\right) \qquad \left(\text{since } \sum_{i=1}^{4} A_i x_i \in T,\ y \in -T^*\right)$$

$$= \sum_{i=1}^{4} \operatorname{Re}(x_i, A_i^H y)$$

> 0, by (I), (II) and the definitions of int S_1^* and of int S_2.

Suppose now that (I) is inconsistent. Then

$$\text{(2)} \qquad \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in T \times S_1 \times S_2 \times S_3 \Rightarrow x_1 = 0 \text{ and } x_2 \notin \operatorname{int} S_2.$$

The first conclusion in (2) is rewritten as follows:

For any $z \in C^n$:

$$\text{(3)} \quad \begin{bmatrix} -A_1 & -A_2 & -A_3 & -A_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in (-T) \times S_1 \times S_2 \times S_3 \Rightarrow \operatorname{Re}\left(\begin{bmatrix} z \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) \geqq 0$$

By Theorem 0 this is equivalent to:

The system

$$(4)\qquad \begin{bmatrix} -A_1^H & I & 0 & 0 \\ -A_2^H & 0 & I & 0 \\ -A_3^H & 0 & 0 & I \\ -A_4^H & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y \\ u \\ v \\ w \end{bmatrix} \in (-T^*) \times S_1^* \times S_2^* \times S_3^*$$

is consistent for any $z \in C^n$. For a choice of z with $-z \in \operatorname{int} S_1^*$, the system (4) gives:

$$(5)\qquad \begin{aligned} &y \in -T^*, \quad A_1^H y = -z + u \in \operatorname{int} S_1^* \quad (\text{since } -z \in \operatorname{int} S_1^*,\ u \in S_1^*) \\ &A_2^H y = v \in S_2^*, \quad A_3^H y = w \in S_3^*, \quad A_4^H y = 0 \end{aligned}$$

The consistency of (5) proves that of (II), if the existence of $v \neq 0$ in (5) can be shown. Suppose that no such v exists. Then

$$(6)\qquad \begin{bmatrix} I \\ A_1^H \\ A_2^H \\ A_3^H \\ A_4^H \end{bmatrix} y \in (-T^*) \times S_1^* \times S_2^* \times S_3^* \times \{0\} \Rightarrow A_2^H y = 0$$
$$\Rightarrow \operatorname{Re}(A_2 z, y) \geqq 0 \text{ for any } z \in C^n.$$

(6) is equivalent by Theorem 0 to the consistency of

$$(7)\quad x_0 + \sum_{i=1}^{4} A_i x_i = A_2 z,\ x_0 \in -T,\ x_1 \in S_1,\ x_2 \in S_2,\ x_3 \in S_3,\ x_4 \in C^n \text{ for any } z \in C^n.$$

If z is chosen so that $-z \in \operatorname{int} S_2$ then (7) gives

$$(8)\qquad \begin{aligned} &A_1 x_1 + A_2(x_2 - z) + A_3 x_3 + A_4 x_4 = -x_0 \in T \\ &x_1 \in S_1, \quad x_2 - z \in \operatorname{int} S_2 \ (\text{since } -z \in \operatorname{int} S_2,\ x_2 \in S_2), x_3 \in S_3 \end{aligned}$$

which contradicts the second conclusion in (2). This completes the proof. □

Related results are:

THEOREM 2. *Let T, A_i, S_i $(i = 1,3,4)$ be as in Theorem 1. Then exactly one of the following two systems is consistent.*

(I) $A_1x_1 + A_3x_3 + A_4x_4 \in T, \quad 0 \neq x_1 \in S_1, \quad x_3 \in S_3$

(II) $y \in -T^*, \quad A_1^H y \in \operatorname{int} S_1^*, \quad A_3^H y \in S_3^*, \quad A_4^H y = 0$

Proof. Delete A_2, S_2, x_2 from the proof of Theorem 1, and follow that proof until (5) which completes the present proof. □

THEOREM 3. *Let T, A_i, S_i $(i = 2,3,4)$ be as in Theorem 1. Then exactly one of the following two systems is consistent.*

(I) $A_2x_2 + A_3x_3 + A_4x_4 \in T, \quad x_2 \in \operatorname{int} S_2, \quad x_3 \in S_3$

(II) $y \in -T^*, \quad 0 \neq A_2^H y \in S_2^*, \quad A_3^H y \in S_3^*, \quad A_4^H y = 0$

Proof. Similarly delete A_1, S_1, x_1 from the proof of Theorem 1, and adapt that proof. □

Some consequences of these theorems are:

COROLLARY 1. (Slater [16])
Let

$$A_i \in R^{mxn_i}, \qquad (i = 1,\cdots,4), \; A_1 \neq 0, \quad A_2 \neq 0$$

Then exactly one of the following two systems is consistent.

$$\text{(I)} \quad \sum_{i=1}^{4} A_i x_i = 0, \quad \left\{ \begin{matrix} x_1 \geqq, \quad x_2 \geqq 0 \\ \text{or} \\ x_1 \geqq 0, \quad x_2 > 0 \end{matrix} \right\}, \quad x_3 \geqq 0$$

(II) $A_1^T y > 0, \quad A_2^T y \geqq 0, \quad A_3^T \geqq 0, \quad A_4^T y = 0$

Proof. Take everything in Theorem 1 to be real with $T = \{0\}$ and $S_i = R_+^{n_i}$, $(i = 1,\cdots,4)$. □

COROLLARY 2. (Motzkin [13], [19])
Let

$$A_i \in R^{mxn_i}, \qquad (i = 1,3,4), \quad A_1 \neq 0.$$

Then exactly one of the following two systems is consistent.

(I) $A_1x_1 + A_3x_3 + A_4x_4 = 0, \quad x_1 \geqq 0, \quad x_3 \geqq 0$

$$A_1^T > 0, \quad A_3^T y \geqq 0, \quad A_4^T y = 0$$

Proof. Similarly follows from Theorem 2. □

COROLLARY 3. (Tucker [18], [19])
Let

$$A_i \in R^{m \times n_i} \quad (i = 2,3,4), \quad A_2 \neq 0.$$

Then exactly one of the following two systems is consistent.

(I) $A_2x_2 + A_3x_3 + A_4x_4 = 0, \quad x_2 > 0, \quad x_3 \geqq 0$

$$A_2^Ty \geqq 0, \quad A_3^Ty \geqq 0, \quad A_4^Ty = 0$$

Proof. Similarly follows from Theorem 3. □

Taking $A_3 = A_4 = 0$ in Corollaries 2 and 3 gives the transposition theorems of Gordan [6] and Stiemke [17] respectively.

These transposition theorems were generalized to the complex case by Mond and Hanson [11], [12]. The following notations and observations are needed to cite one of their results:

For $\alpha \in R^n_+$: $\alpha \leqq \pi/2$ denotes $\alpha_i \leqq \pi/2 \quad (i = 1,\cdots,n)$

For $\alpha \in R^n_+$, $z \in C^n$:

$$|\arg z| \leqq \alpha \text{ denotes } |\arg z_i| \leqq \alpha_i \quad (i = 1,\cdots,n)$$

For $\alpha \in R^n_+$, $\alpha \leqq \pi/2$, $S = \{z : |\arg z| \leqq \alpha\}$ is a polyhedral cone and its polar is $S^* = \{z : |\arg z| \leqq \pi/2 - \alpha\}$, e.g. [3], example 1.2(e).

COROLLARY 4. (Mond and Hanson [11])
Let

$$A_i \in C^{m \times n_i} \quad (i = 1,3,4), \; A_1 \neq 0, \quad \alpha \in R^n_+, \quad \alpha \leqq \frac{\pi}{2}.$$

Then exactly one of the following two systems is consistent.

(I) $A_1x_1 + A_3x_3 + A_4x_4 = 0, \quad \operatorname{Im} x_1 = 0, \quad \operatorname{Re} x_1 \geqq 0, \quad |\arg x_3| \leqq \alpha$

(II) $\operatorname{Re} A_1^Hy > 0, \; |\arg A_3^Hy| \leqq \frac{\pi}{2} - \alpha, \quad A_4^Hy = 0$

Proof. Follows from Theorem 2 with

$$T = \{0\}, \quad S_1 = R^{n_1}_+ \text{ (in } C^{n_1}), \quad S_3 = \{z : |\arg z| \leqq \alpha\}.$$ □

The other complex transposition theorems of Mond and Hanson [12] similarly follow from the above theorems.

2. **Remarks.** (i) The following (real) example shows that Theorem 2 cannot be extended to general (non polyhedral) closed convex cones.

Let

$m = 3$

T = all vectors in R^3 forming an angle $\leqq 45°$ with $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$n_1 = 1, \quad S_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad S_1 = R_+ \text{ (the nonnegative reals)}$$

$$n_3 = 3, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_3 = \text{all vectors in } R^3 \text{ forming an angle}$$

$$\leqq 45° \text{ with} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then neither (I) nor (II) of Theorem 2 are consistent.

(ii) The solvability Theorem 0 can easily be shown to follow from Theorem 2. Thus Theorems 0, 1, 2 and 3 are equivalent. See also [1] where the equivalence of Corollaries 2 and 3 is proved.

(iii) The above theorems of the alternative and transposition theorems are sometimes more convenient in applications than the (logically equivalent) solvability theorems such as [5], [4] or Theorem 0 above. For applications of transposition theorems see for example [10], [12], [14], [15] and [19].

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